AtCoder Peterzavodsk Contest 001 解説

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For International Readers: English editorial starts on page ?.

A: Two Integers

Y が X の倍数な場合、 X の倍数はかならず Y の倍数になってしまいます。なのでこの場合は -1 を出力します。

そうでない場合、X を出力すればよいです。

B: Two Arrays

まず，操作をすると，$\sum a_i$ は 2 増え，$\sum b_i$ は 1 増えます。つまり，操作回数は $\sum b_i - \sum a_i$ 回です。

- $a_i > b_i$ の場合，少なくとも b_i を $a_i - b_i$ 回，1 増やす必要があります。
- $a_i < b_i$ の場合，少なくとも a_i を $\lceil (b_i - a_i)/2 \rceil$ 回，2 増やす必要があります。

実はこの 2 つが必要十分です。つまり，この 2 つの式で必要最低限の 1 加算，2 加算を計算し，それが操作回数より多くなければ答えは YES です。

なお，実は 2 つめの条件だけで十分なことが示せます。

C: Vacant Seat

必ず空席を含むような区間を二分法によって狭めていくことで，$O(\log N)$ 回のクエリで空席を見つけることができます．必ず空席を含むような区間というのは，次の 2 通りです:

(a) 閉区間 $[l, r]$ であって，$r - l$ が奇数で，席 l, r が同性であるもの．
(b) 閉区間 $[l, r]$ であって，$r - l$ が偶数で，席 l, r が異性であるもの．

(a) に空席が含まれないと仮定すると，席 $l, l+1, \ldots, r$ には男性と女性が交互に座ることになり，席 l, r は異性となって矛盾します．(b) も同様です．

二分法は次のように行います:

- まず，席 0, $N-1$ を質問する．いずれかが空席ならば終了．そうでなければ，区間 $[0, N-1]$ を得る．
- 区間 $[l, r]$ に注目しているとき，席 $m := \lfloor l + r \rfloor / 2$ を質問する．空席ならば終了．そうでなければ，区間 $[l, m-1]$, $[m+1, r]$ の一方は (a), (b) のいずれかに該当するので，そちらを新しい区間とする．

D: Forest

コーナーケース: $M = N - 1$ の場合，元から木なので答えは 0 です。

以下では，$M < N - 1$ を仮定します。

まず，深さ優先探索などを使い，入力された森を木に分解します。

この木たちをくっつけて 1 つの巨大な木にするわけですが，この時，以下の 2 つの性質がわかります。

- どの木からも，少なくとも 1 つの頂点は選択される- 連結成分の個数は $N - M$ 個である。つまり，$N - M - 1$ 本の辺が足される。よって，全て合わせて $2(N - M - 1)$ 個の頂点が選ばれる。

2 つめの性質より，$N < 2(N - M - 1)$ ならば答えは 0 です。

ところで，実はこの 2 の性質は必要十分条件です。つまり，頂点から

- 連結成分ごとに，少なくとも 1 つの頂点を選ぶ
- 全て合わせて $2(N - M - 1)$ 個の頂点を選ぶ

の 2 つの条件を選ぶように頂点を選んだ時，また，その時のみ，選んだ頂点のみを使って森を木にすることができます。

実際に，最も選ばれた頂点数が多い 2 の連結成分をマージする，という戦略を繰り返すことで，かならず 1 つの木にすることができます。

よって，まず各連結成分から 1 番小さい値を選び，残ったものすべてから小さい順に，合計 $2(N - M - 1)$ 個になるように選べば，それが答えです。

これは，優先度付きキューや sort などをうまく使えば $O(N \log N)$ で行うことができます。

E: Antennas on Tree

証明は省きますが，問題の条件は次と同値です:

> 木の各頂点 v について，次が成り立つ: 木から v を取り除くと，木が k 個の部分木に分かれるとする．これらのうち $k-1$ 個以上の部分木はそれぞれアンテナを含む．

木のすべての頂点の次数が 2 以下の場合，木はパスの形をしており，いずれかの端にアンテナを置けばよいので，答えは 1 です．以降，木は次数 3 以上の頂点を含むとします．次数 3 以上の頂点 r をひとつ選び，r を根とする根付き木を考えます．すると，上述の条件は次と同値です:

> 根付き木の各頂点 v について，次が成り立つ: v が k 個の子を持つとする．これらをそれぞれ根とする k 個の部分木を考える．これらのうち $k-1$ 個以上の部分木はそれぞれアンテナを含む．

r 以外の各頂点 v について，v の親側の部分木には必ずアンテナが含まれます．なぜならば，r の子を根とする部分木のうちアンテナを含むものは 2 個以上ありますが，これらのうちひとつは v の親側の部分木に完全に含まれるからです．

r を根とする根付き木に対して，上述の条件が成り立つようなアンテナの個数の最小値を求めます．これは DP で可能です．各頂点 v と $x \in \{0,1\}$ について，$dp[v][x]$ を「v を根とする部分木内で，上述の条件が成り立ち，アンテナが存在しない $(x=0)$ / 存在する $(x=1)$ 場合の，アンテナの個数の最小値」と定義し，ボトムアップに計算すればよいです．

F: XOR Tree

まず各頂点 v に対して、v と接続する全ての辺 e に対し a_e を xor した値を b_v と定義します。すると、頂点 u,v を選んでパス上の辺の a_e に x を xor するという操作は、 b の言葉に言い直すと、b_u と b_v に x を xor するという操作になります。また、全ての e に対して a_e が 0 であることと、全ての v に対して b_v が 0 であることは同値です。従って、以下のような問題に置き換えられます。

$b_1, b_2, .., b_N$ が与えられる。i, j, x を選んで b_i と b_j に x を xor するという操作ができる。最小何回の操作で全てを 0 に出来ますか?

新しく以下のようなグラフを考えます。

- 頂点集合は $1, 2, .., N$
- 頂点 u, v を選んで操作した場合、u, v に辺を張る。

操作列の頂点のペアだけ先に決めた時に x を上手く決めて目標を達成できるかを考えます。まず各連結成分ごとに考えても構いません。各連結成分に対し、その頂点集合の b の値の xor が 0 でない場合、これは不可能です。逆に、0 の場合、これは可能であり、操作回数は、(その連結成分の頂点数) -1 になります。

従って、操作回数を最小化するためには、$\{1, 2, .., N\}$ を、できるだけ多くの、b_v の xor が 0 になるような部分集合に分割することができれば良いです。

証明は省略しますがこれは以下のようにできます。

全ての $k \geq 0$ に対し、$b_v = k$ なる v の個数を c_k とします。

まず $b_v = 0$ なる v に対しては単独で取れば良いです。これで c_0 個の部分集合がとれます。

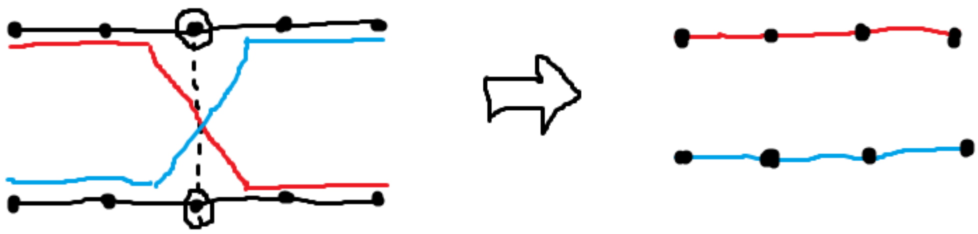
次に、$k > 0$ に対し、$b_v = k$ なる v 同士で出来るだけペアを作ります。これで $\lfloor c_k/2 \rfloor$ 個の連結成分がとれます。

すると最後に残るのは 1 から 15 の値が高々 1 つずつなので、これは bit DP で $O(3^{15})$ で出来ます。

G: Colorful Doors

ドア $2N$ のすぐ右にドア 1 があると考えると，全体が円環となります．このとき，ドア間の区間は全体で $2N$ 個あり，ドア $2N, 1$ 間の区間は歩くべき区間です．

- すべての区間が歩くべき区間である場合
  この場合，歩くべき区間はひとつのサイクルになります．
  - (i) $2N$ が $4k$ の場合
    時計回りにドアの色を $1, 2, 1, 2, 3, 4, 3, 4, 5, 6, 5, 6, \ldots, 2N-1, 2N, 2N-1, 2N$ とすればよいです．
  - (ii) $2N$ が $4k+2$ の場合
    証明は省きますが，不可能です．
- 歩くべきでない区間が存在する場合
  この場合，歩くべき区間はいくつかのパスの集合になります．両側を歩くべき区間に挟まれているドアを「内部のドア」と呼ぶことにします．各パスの「長さ」を，そのパスに含まれるの内部のドアの個数と定義します．例えば，$s =$ `010110011` の場合，パスの長さはそれぞれ $0, 1, 2$ です (末尾に `1` を追加した上で円環にしていることに注意)．
  - (iii) パスの長さの総和が $2k+1$ の場合
    不可能です．なぜならば，内部のドアは別の内部のドアとマッチングさせる必要があるので，内部のドアの総個数は偶数でなければならないからです．
  - (iv) パスの長さの総和が $4k$ の場合
    次のようにして可能です．パスの端どうしを順に繋げていくと，ひとつのサイクルになります．このサイクルは内部のドアを 4 の倍数個含みます．以降はケース (i) と同じです．
  - パスの長さの総和が $4k+2$ の場合
    * (v) 長さ 1 以上のパスが 2 本以上ある場合
      長さ 1 以上のパスを 2 本選び，それらの長さを x, y とします．各パスから内部のドアを 1 個ずつ選び，それらをマッチングさせます．すると，下図のようにパスが組み替えられ，これらの長さの和は $x + y - 2$ となります．以降はケース (iv) と同じです．



    * (vi) 長さ 1 以上のパスが 1 本の場合
      長さ $4k+2$ のパスが 1 本と，長さ 0 のパスが何本かあります．これらのパスの端どうしを順に繋げるしかなく，ケース (ii) に帰着されて不可能です．

H: Generalized Insertion Sort

まず，頂点ごとに値の書かれたボールが入ってるとします。

もし葉に正しいボールが入ってたら，その頂点をカットすることが出来ます。これで葉からカットしていって，根まで全部カットする，というのが基本方針です。

ですが，葉を 1 個ずつカットしていてはクエリがとても足りないです。なので，葉パス頂点を，

- 頂点が葉である
- 子が 1 個だけであり，その子が葉パス頂点である

として定義します。

すると，高々 $O(\log n)$ 回「木の葉パス頂点をカットする」を繰り返すと，木を全部カットすることが出来ます。これは，k 回のカットに耐えるためには $k-1$ 回のカットに耐える子を 2 個は持たないといけないことから示せます。

よって，$O(n)$ 回で，木の葉パス頂点に正しい値のボールを全部突っ込む，というのが出来ればこの問題は $O(n \log n)$ 回で解けます

これは，葉パス頂点に入れるべきボールを全部赤く塗って，その他を白く塗って，

- 根に赤いボールが出てきたら，対応する頂点の葉パスの一番下から頂点を見ていって，「自分より入るべきところが上 || !青いボール」が出てきたら，そこにボールを入れて，ボールを青く塗る
- 根に白いボールが出てきたら，赤いボールのうち最も下のものを選んで，そこに入れる

というアルゴリズムで行なえます。

具体的にクエリ回数を見積もると，カット回数が 11 で，各フェーズでは (全体数 + 葉パス頂点の数) でカットできるので，24000 回で抑えることが出来ます。

なお，実は $O(n)$ 回のクエリでも可能な解法があります (by rng_58)。具体的な解法の説明は省略しますが，時系列を逆転させ，バランスのいいほうき (長いパスの末尾から木が生えている形) を探し，木を消していくのを繰り返す，という方針です。

I: Simple APSP Problem

制約より，かなり黒いマスは「疎」であることがわかり，またこの性質を使うのだろうと想像がつくと思います。

まず，すべてのマスが白い行や列が大半を占めることがわかります。ではここで，i 行目と $i+1$ 行目のマスが全て白いと仮定します。

すると，i 行目側と $i+1$ 行目側から 1 個ずつ白いマスを選んだ時，またその時のみ，最短パスは i 行目と $i+1$ 行目の間を移動する事がわかります。

つまり，i 行目と $i+1$ 行目の間の辺の距離を全て 0 にしたとすると，この問題の答えは (i 行目側の白いマスの個数) $\times$ ($i+1$ 行目側の白いマスの個数) だけ減ることがわかります。

実は，この性質に気づくとこの問題は殆ど解けています。2 行連続で黒いマスが存在しないとき，その間を縮約することが可能なので，列に対しても同様の処理を行うと，この問題は高々 $H, W \leq 2N$ のグリッドに帰着できることがわかるからです。

ただし注意点として，元の問題とは違い，各マスごとに複数のマスが圧縮されていることがあります。ですので係数に気をつける必要がありますが，各マスから愚直に幅優先探索を行うことで，$O(N^4)$ で解くことが可能です。

J: Rectangles

まず A が a で割り切れない場合、B が b で割り切れない場合、C が c で割り切れない場合は条件を満たすものはありません。

とある平面で切れるようなものだけを考えると、どの平面で切れるかで包除をするとその個数は求まります。ただしこれはまだ答えではありません。つまりどんな平面でも切れないようなケースが存在します。

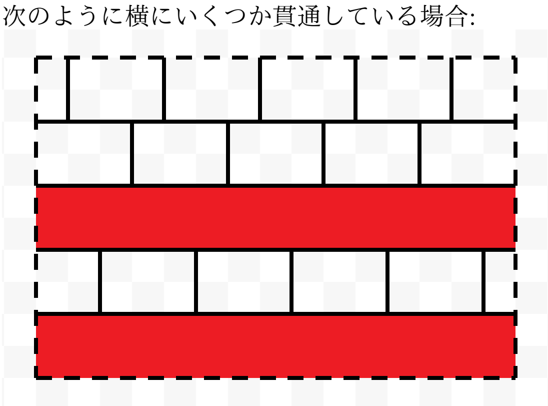
まず条件を満たすトーラス直方体の集合が与えられたとします。このとき、p, q, r を用いて書いたトーラス直方体 $\{((p+i) \mod A, (q+j) \mod B, (r+k) \mod C))\}$ に対して、その各小立方体に値 k を書き込みます。これによって全小立方体に値が書き込まれますが、マス (x, y, z) に書き込まれた値を $v(x, y, z)$ とします。このとき、各 x, y に対して、$v(x, y, 0), v(x, y, 1), .., v(x, y, C-1)$ という列を考えると、これは $0, 1, .., c-1, 0, 1, .., c-1, .., 0, 1, .., c-1$ というものをサイクリックに動かしたものになっています。従って、$v(x, y, 0)$ の値を決めれば $v(x, y, z)$ の値は全て決まります。なので以降は $v(x, y, 0)$ のみを考えることにし、これを $v(x, y)$ とよぶことにします。

全てのペア $0 \leq x < A, 0 \leq y < B$ に対して $v(x, y)$ が定まっている時、この条件をみたすトーラス直方体の集合が何通りあるか考えます。これは、全ての $0 \leq k < C$ に対して、$v(x, y) = k$ をみたす (x, y) の集合を取ってきて、それをサイズ $a \times b$ のトーラス長方形に分割する通り数の積 を C/c 乗したもの となります。

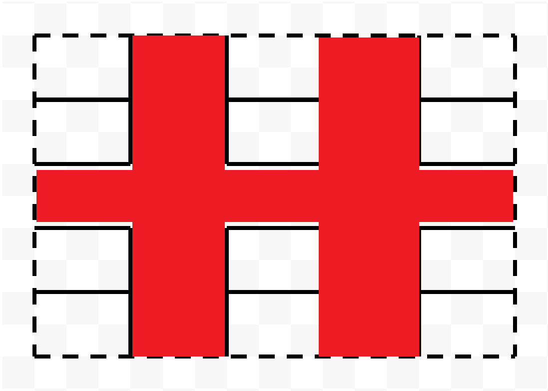
よってまず (x, y) の集合 S が与えられた時に、それをサイズ $a \times b$ のトーラス長方形に分割する通り数をカウントすることを目標にします。

次のようなケースが考えられます。

1. $S = \emptyset$
2. $S = \{(x, y) | 0 \leq x < A, 0 \leq y < B\}$
3. 次のように横にいくつか貫通している場合:



4. 縦に貫通している場合:
5. 縦横両方に貫通している場合:



6. それ以外:

ケース 5 となるような k が存在する場合、そのような k はひとつしか存在しません。このとき縦に i 本, 横に j 本貫通している時の求めたい値は除原理で $O(100^4)$ で求めることが出来ます。

ケース 5 となるような k が存在しない場合を考えます。すると必ずケース 2, 3, 4 となる k が存在することが示せます。これらの場合はどれも、とある平面で切れるケースとなっています。よって、事前にとある平面で切れるケースの通り数を求めておけばここでは数えないことにすれば良いです。

AtCoder Petrozavodsk Contest 001 Editorial

japan02

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A: Two Integers

If X is a multiple of Y, the answer is -1: a multiple of X is always a multiple of Y. Otherwise, X is always a valid answer.

B: Two Arrays

In each operation, the value $\sum b_i - \sum a_i$ decreases by 1. Thus, the total number of operations must be $K = \sum b_i - \sum a_i$.

In case $a_i < b_i$ for some i, you must perform operations on a_i at least $\lceil (b_i - a_i)/2 \rceil$ times. Thus, the sum of $\lceil (b_i - a_i)/2 \rceil$ for all i such that $a_i < b_i$ must be at most K. Otherwise, print "NO".

On the other hand, when this condition is satisfied, we can prove that the answer is "YES". Notice that the order of operations of adding twos and ones doesn't matter. First, add 2 to a_i, $\lceil (b_i - a_i)/2 \rceil$ times. Then, perform remaining "add two" operations on arbitrary elements. At this point, all i satisfies $a_i \geq b_i$, so you can finish operations by performing "add one" operations appropriately.

C: Vacant Seat

Consider the following two types of intervals (we call it "good interval"):

(a) A closed interval $[l, r]$ such that $r - l$ is odd and l, r are filled by people with the same sex.

(b) A closed interval $[l, r]$ such that $r - l$ is even and l, r are filled by people with different sex.

In both cases, we can see that the interval contains at least one empty seat. (Otherwise, males and females must sit alternately in the interval $[l, r]$, and we get a contradiction in both cases.)

We use binary search, and find an empty seat in $O(\log N)$ queries.

- First, perform a query on seat 0. If this is empty, we finish. Otherwise we find a good interval $[0, N]$. (Since the seats are cyclic, we can call seat 0 "seat N").
- Suppose that we have a good interval $[l, r]$. Let $m := \lfloor l + r \rfloor / 2$ and perform a query on seat m. If this is empty, we finish. Otherwise, we can prove that at least one of intervals $[l, m]$, $[m, r]$ will be a good interval by a simple argument about parities.

D: Forest

If $M = N - 1$, the input is a tree, and the answer is 0. Assume that $M < N - 1$.

First, divide the forest into connected components (trees). We want to merge these trees into a single tree by adding edges.

Here, we have the following constraints:

- From each tree, at least one vertex must be chosen (otherwise this tree will be isolated).
- There are $N - M$ trees. Thus, we must add $N - M - 1$ edges. This means that the total number of chosen vertices must be $2(N - M - 1)$.

It turns out that these conditions are sufficient. That is, if a subset of vertices satisfies the conditions above, we can actually make a tree by choosing those vertices. For example, we can connect all trees by repeating the following: choose two components with the biggest number of chosen (and still unused) vertices, and connect them.

The answer is as follows. First, if $N < 2(N - M - 1)$, the answer is "Impossible" (from the second condition). Otherwise, we choose vertices greedily as follows:

- First, from each component, choose a vertex with the smallest cost.
- Then, from remaining vertices, choose vertices from the smallest costs, until we choose $2(N-M-1)$ vertices in total.

These operations can be implemented in $O(NlogN)$ time.

E: Antennas on Tree

Consider two vertices s, t in the tree. When can we distinguish these vertices?

If the distance between them is odd, we can always distinguish them as long as we have at least one antenna (by checking the parity of distance from the antenna). Otherwise, let u be the midpoint between s and t. We see s, t and antennas from u. If the direction of at least one antenna is the same as the direction of s or t, we can distinguish them.

Thus, the condition can be restated as follows:

> For each vertex v, the following holds. Remove v from the tree, and suppose that we get k subtrees. Then, at least $k - 1$ of the subtrees must contain antennas.

Now, we'll describe the solution. In case the tree is a path, the answer is one: we can put an antenna on one of the leaves. Otherwise, a vertex r with degree at least 3 exists. Make it the root of the tree.

The condition can be again restated as follows:

> For each vertex v in the rooted tree, the following holds. If v has k children, at least $k - 1$ of k subtrees whose roots are children of v must contain antennas.

This is because, if v is not r, v always contain at least one antenna in the "parent direction". (Otherwise, the condition above won't be satisfied for the root r).

Now we can compute the answer by a simple DP. Define $dp[v]$ as the smallest number of vertices we must choose from the subtree rooted at v, when we want to satisfy the conditions above for all vertices in this subtree.

F: XOR Tree

For a vertex v, define b_v as the XOR of all values assigned to edges incident to v. If we perform an operation on the path between two vertices u and v with the value xm we change b_u and b_v to $b_u XOR x$ and $b_v XOR x$, respectively. Also, note that all b_v will be zero if and only if all edges are assigned zeroes. Thus, the problem can be restated as follows:

> You are given a sequence of integers $b_1, b_2, .., b_N$ (here, all integers are up to 15). In each operation, you choose i, j, x, and XOR x into b_i and b_j. How many operations do you need to make all integers zeroes?

Now, consider a graph with N vertices $1, 2, .., N$. Initially, this graph doesn't have edges. Whenever you perform an operation for indices i and j, add an edge between vertices i and j.

After we perform all operations, this graph will have several connected components. When you perform an operation between i and j, the value $b_i XOR b_j$ doesn't change. Thus, the XOR of all values in a single connected component never changes.

Therefore, we want to add minimum possible number of edges to this graph such that in each connected component, the XOR of all b_i is zero. Obviously, each connected component should be a tree if we want to minimize the number of edges. In this case, the total number of edges is N minus the number of connected components.

Thus, we can again restate the problem as follows:

> You are given a sequence of integers $b_1, b_2, .., b_N$ (here, all integers are up to 15). Divide these numbers into disjoint set, such that in each set the XOR of all values is zero. The answer is N minus the maximum possble number of sets we get.

First, we should make sets as follows:

- If we have a zero, we should create a set with this single zero.
- If we have two identical numbers, we should create a set with these two numbers.

After these process, we will have at most 15 elements. Now, we can compute the maximum number of sets in a simple $O(3^k)$ DP, where k is the number of remaining elements.

G: Colorful Doors

Make the segment cyclic, that is, assume that door 1 is to the right of door $2N$. Now we have a circle, there are $2N$ doors on the circle, and there are also $2N$ section between two doors. The new section between $2N$ and 1 should be walked through.

Under this setting, it's easier to prove the claim from the statement: "It can be shown that he will eventually get to the right bank". Choose an arbitrary section (call it section s), and start walking from section s (keep walking forever). Since the number of sections is finite, he will eventually pass through the same section twice. So, the sequence of sections we pass through will be periodic from some point: "$s, \cdots, t, \cdots, t, \cdots, t, \cdots,$". Furthermore, since the section we pass through immediately before a certain section can be uniquely determined, the entire sequence will be periodic: "$s, \cdots, s, \cdots, s, \cdots, s, \cdots,$". This proves the claim from the statement because if we start from section 0, we will return to this section in the future.

By the same observation, we can divide the $2N$ sections into one or more cycles. We want to find a coloring of doors such that the set of sections represented by '1' in the input corresponds to one of the cycles.

- In case all sections should be walked through (i.e., all sections are in the same cycle)

  - (i) In case $2N = 4k$ for some k, the sequence of doors $1, 2, 1, 2, 3, 4, 3, 4, 5, 6, 5, 6, \ldots, 2N - 1, 2N, 2N - 1, 2N$ is a valid answer.
  - (ii) In case $2N = 4k + 2$, we can prove that the answer doesn't exist. Let's start from the case $N = 1$. In this case the number of cycles is two. After that, increase N one by one. In each step, we can see that the parity of the number of cycles always change, regardless of the positions of two new doors we add. Thus, when $2N = 4k + 2$, we have even number of cycles, and all sections can't be in one cycle.
- Other cases

  Now, the set of '1's will be a set of two or more paths. Let's define the length of a path as the number of doors in the path, excluding two doors at the ends. For example, if $s =$ `010110011`, after we make it cyclic we get $s =$ `1010110011`, and the set of lengths of paths is $0, 1, 2$. It's easy to see that only this set matters: the order of these paths or the number of '0's between them don't matter. Now, we consider several cases depending on the sum of these lengths.

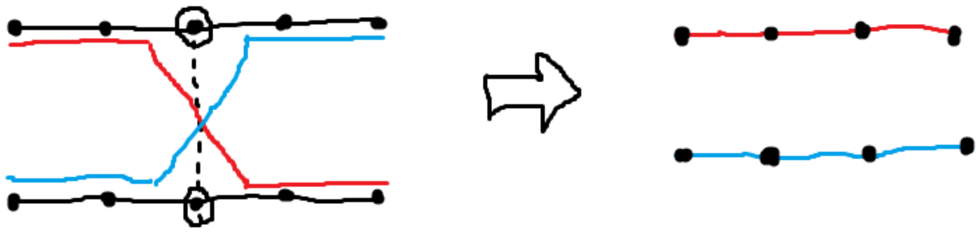
  Let's call a door "internal" if it's between '1' and '1', "external" if it's between '0' and '0', and "boundary" otherwise. An internal door must be passed through in both directions, a boundary door must be passed through in one direction, and an external door must not be passed through.

  - (iii) The sum of these lengths is odd

    From the observation above, an internal door must be matched with another internal door. However, in this case, the total number of internal doors is odd, and this is impossible.
  - (iv) The sum of these lengths is $4k$

Suppose that the set of lengths contains $\{x\}$ and $\{y\}$. By connecting the first section of $\{y\}$ immediately after the last section of $\{x\}$, we can regard them as $\{x + y\}$. By repeating this, we will eventually have a single element divisible by 4, and it reduces to the case (i).

– The sum of these lengths is $4k + 2$ - now we have two cases:
  * (v) We have at least two paths with nonzero lengths
    Suppose that the set of lengths contains two nonzero elements $\{x\}$ and $\{y\}$. Choose one internal door from each path and match them. Then, these two paths can be regarded as two paths whose sum is $x + y - 2$ (see the picture below). It reduces to the case (iv).



  * (vi) All paths but one have lengths of zero
    All paths of lengths zero must be merged together, and this reduces to the case (ii). This is impossible.

H: Generalized Insertion Sort

Assume that each vertex contains a ball, and each ball has an integer written on it. We move balls by operations described in the statement.

Here's our plan. First, we choose an arbitrary leaf, and perform some operations until we assign a correct ball to the chosen leaf. After that, we can ignore this leaf (never perform operations that affect this leaf). By repeating this process from leaves to roots, we can achieve the goal.

To reduce the number of operations, we process some leaves (and other vertices) at once, as follows. We define "leafish vertex" as follows:

- A leaf is a leafish vertex.
- If a vertex has exactly one child, and the child is leafish, is parent is also leafish.

Each time we remove all leafish vertices, the total number of leaves is always halved. It means that we can make the tree empty by repeating the process "remove all leafish vertices "$O(logn)$ times. Thus, if we can assign correct balls to all leafish vertices in $O(n)$ operations, we can achieve the goal in $O(nlogn)$ operations.

How do we do that? Let's color balls. We color the balls that should be assigned to leafish vertices red. Other balls are white.

Then, repeat the following:

- If the ball at the root is red, insert it into a "correct position", and color it red. More specifically, look at the path that contains the destination of this ball. Check balls in the path from bottom to top. Whenever we find a non-blue ball or a blue ball that should be above our current ball in the final position, insert our current ball there.
- Otherwise, call the ball at the root B. Then, find the bottommost ball that is red or white: call it C. Paint B black, and insert it to the position of C.

Here, blue balls represent balls that are assigned to the correct path, in "correct order". At each moment, all blue balls are in leafish vertices, and inside each path, the blue balls are below other balls. Furthermore, the relative order of blue balls are the same as there order in the final position.

How many operations do we need for this process? Since we color red and white balls after the very first operations, the total number of operations for those balls is at most n. Blue balls never appear at the root because all of them are at leafish positions. Black balls may reach the root only after we perform an operation for red balls. Thus, the total number of operations for black balls is at most (the number of leafish vertices), and the total number of operations for all operations is $n+$ (the number of leafish vertices).

The total number of phases is at most 11, and in each phase the number of operations is at most $n+$ (the number of leafish vertices). Thus, we need at most 24000 operations.

By the way, if you want more efficient solution, we can actually do it in $O(n)$ operations. Here's a brief sketch of the solution:

- See operations in the reverse order: we insert a chosen ball into the root.
- We define "broom", a set of vertices that satisfies the following properties. Let x be an arbitrary vertex in the tree, and $c_1, \cdots, c_k$ be a subset of children of x. A broom consists of two parts: a "path part" and a "tree part". The path part is simply a path between the root and x. The tree part is the union of subtrees rooted at $c_1, \cdots, c_k$.
- Consider a broom with p vertices in the path part, and q vertices in the tree part. We call it "balanced" if $2q - c \leq p \leq 4q + c$ (where c is a small constant).
- Prove that a tree always have a balanced broom.
- Take a broom from a tree. Paint all q balls that should be assigned to the tree part of the broom red. Find a way to put all red balls into the path part of the broom, in $O(p + q) = O(q)$ steps.
- Suppose that now all red balls are in the path part of the broom. Find a way to put all red balls into correct positions in $O(p + q) = O(q)$ steps.
- Now, just repeat this and we achieve the goal.

I: Simple APSP Problem

The black cells will be very sparse, and the vast majority of rows and columns will be entirely white.

Suppose that two adjacent rows, the i-th row and the $i+1$-th row, are entirely white. Let's call the region in the i-th row and above "UP", and the $i+1$-th row and below "DOWN". It's easy to see that the shortest path between two cells crosses an edge (here, we imagine that white cells correspond to vertices and there are edges for each adjacent pair of white cells) between the i-th row and the $i+1$-th row if and only if one of the cells is UP and the other is DOWN.

Thus, if we change the costs of all edges between the i-th row and the $i+1$-th row to zeroes, the answer decreases by the following value:

(The number of white cells in UP region) $\times$ (the number of white cells in DOWN region)

Now, we can "compress" two adjacent empty rows, and we can do the same for columns. By repeating these operations, we can compress the entire grid into a smaller grid with $H, W \leq 2N$, and we can simply solve the problem by BFS in $O(N^4)$.

Note that we should add "weights" to cells after compressions. For example, in the very first compression, if we merge the i-th row and the $i+1$-th row into a single row, the cells of this row should have a weight of 2. Then, in the final grid, the distance between two cells p and q should be added to the answer with the coefficient (the weight of p) $\times$ (the weight of q).

J: Rectangles

The 2D version of the problem is easy. Consider two adjacent rectangles. If these two rectangles are not "aligned" well, we can uniquely determine the positions of more rectangles, and we will eventually fill a entire row or a column. Thus, there is a line parallel to one of coordinate axis that doesn't split any rectangles. It's not hard to count such patterns.

Now, let's solve the original 3D problem. Assume that A, B, C are multiples of a, b, c, respectively (otherwise the answer is zero). We call a pattern "trivial" if there is a plane that doesn't split any cuboids. We can count the number of trivial patterns easily. The number of patterns that can be cut by a plane can be reduced to the 2D case. Do not forget to avoid double-counting by using inclusion-exclusion principle. For example, you should subtract the number of patterns that can be cut by planes of two directions.

The main challenge is that, in 3D case, there are non-trivial patterns. Let's think how these patterns look like.

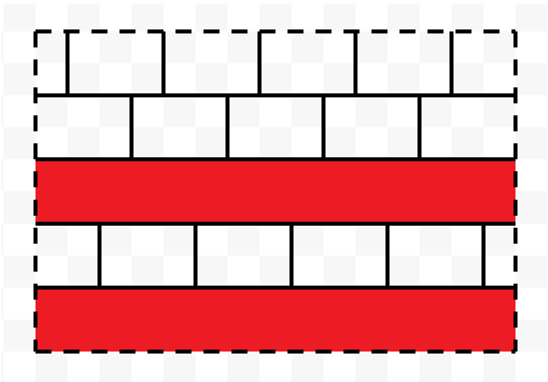
First, for a torus cuboid with parameters (p, q, r) in the statement, write an integer k into the small-cube $\{((p+i) \mod A, (q+j) \mod B, (r+k) \mod C))\}$. This way all small-cubes will contain an integer. Let $v(x, y, z)$ be the integer written on (x, y, z) For each pair (x, y), the sequence $v(x, y, 0), v(x, y, 1), \cdots, v(x, y, C-1)$ will be a cyclic shift of $0, 1, .., c-1, 0, 1, .., c-1, .., 0, 1, .., c-1$. Thus, the values of $v(x, y, 0)$ determines all values of v. Let $v(x, y) = v(x, y, 0)$, and consider an $A \times B$ table whose (i, j)-element is $v(i, j)$.

Now, for a given table, we want to count the number of patterns that are consistent with this table. Fix a pattern that is consistent with the table. In each layer (here a layer means a plane with constant value of z-coordinate), we get a partition of the entire $A \times B$ rectangle into $a \times b$ torus rectangles that corresponds to the pattern. Here, notice that a torus rectangle never contains two cells with different values of v, and the partition at the layer with $v(x, y, z) = 0$ determines everything else.

Thus, we get the following. Let $f(h)$ be the number of ways to partition the set of cells (x, y) such that $z(x, y) = h$ into torus rectangles. Then, the number of patterns that is consistent with the table is $f(0)^{C/c} \times \cdots \times f(c-1)^{C/c}$.

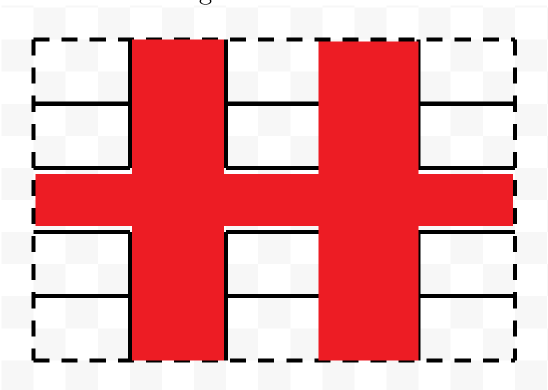
Now, fix an $A \times B$ table (with the values of v). When do we have non-zero number of non-trivial patterns that is consistent with this table?

We call this table "row-aligned" if in each row, all numbers are the same. Similarly, define "column-aligned". If the table is both row-aligned and column-aligned, it means that the table only contains a certain constant, and this corresponds to patterns that can be cut by xy-plane. Since this pattern is trivial, we can ignore it. If the table is row-aligned (or similarly, column-aligned), the table is multiple stripes whose heights are multiples of a (see the picture below). In this case the only way to partition it into torus rectangles is to entirely divide it into stripes with heights a, thus again it corresponds to a trivial pattern. Thus, we assume that this table is not aligned in any directions.



Consider a way to partition this table into torus rectangles of dimensions $a \times b$. Each torus rectangle must contain the same values of v. As we see in the 2D case, this partition is either "horizontal" (i.e., a union of stripes of height h, some stripes are possibly shifted horizontally), or "vertical". If all such ways are "horizontal" (or "vertical"), it corresponds to a trivial pattern. Thus, there must be both horizontal ways and vertical ways to partition it. It means that the entire $A \times B$ table must be splitted into $a \times b$ torus rectangles in the most natural way (i.e., all rectangles are aligned like a grid). Also, since the table is not aligned, there is a unique way to do so. Let's call it "standard partition".

In order for the pattern to be non-trivial, in at least one layer the partition must be shifted to horizontal direction, and in at least one layer the partition must be shifted to vertical direction. This means that there exists an integer h that dominates some rows and some columns, as in the picture below:



Since we have a standard partition, now we can consider the entire table as an $A/a \times B/b$ table. As we see above, there are h, p, q such that h dominates exactly p rows ($0 < p < A/a$) and q columns ($0 < q < B/b$). In this case, the number of standard partition is 1, the number of horizontally shifted partitions is $b^p - 1$, and the number of vertically shifted partitions is $a^q - 1$. We want to count the number of ways to choose a sequence of C/c partitions such that at least one is horizontally shifted, and at least one is vertically shifted. This value is $(b^p + a^q - 1)^{C/c} - (b^p)^{C/c} - (a^q)^{C/c} + 1$.

To summarize, the solution is as follows:

- Count the number of trivial patterns by inclusion-exclusion.
- Let's count the number of non-trivial patterns. First we fix a standard partition (ab ways) and the value of h (c ways). Then, for each pair (p, q), compute the following values:

- Suppose that we have an $A/a \times B/b$. How many ways are there to fill this tables with integers between 0 and $c-1$, such that exactly p rows are dominated by h and exactly q columns are dominated by h? This can be done by a simple $O(N^4)$ DP ("exclusion principle").
- Compute the value $(b^p + a^q - 1)^{C/c} - (b^p)^{C/c} - (a^q)^{C/c} + 1$.

We compute the product of two values above, compute the sum for all pairs (p, q), and multiply it by a factor of abc.